"ABC" of Relativistic QM

$$E = \frac{p^2}{2m} = \underbrace{\frac{1}{2}mv^2}_{} \text{(free particle)}$$

↑ Newtonian (non-relativistic)

QM: $p \to \hat{p} \to \frac{\hbar}{i}\vec{\nabla}$; $E \to i\hbar\frac{\partial}{\partial t}$

⇒ Schrödinger Equation ... (1925)

$\boxed{\text{Non-relativistic QM}}$

→ Triumph of 20th century physics!

→ Led to: Solid state physics

Semiconductor industry

Electronic devices

Computers

Atomic physics

Molecular physics / Quantum Chemistry

...

- Very early on, physicists (Klein, Gordon (1927)) (Dirac 1928) tried to formulate a Quantum Mechanics that is consistent with special relativity.

Klein-Gordon (1927) did the earliest attempt.

$$E^2 = c^2p^2 + m^2c^4$$

starting point is relativistic!

$E \to i\hbar \frac{\partial}{\partial t}$; $p_x \to \frac{\hbar}{i}\frac{\partial}{\partial x}$, etc

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

$$\underbrace{\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \frac{m^2c^2}{\hbar^2}\psi}_{\text{Klein Gordon Equation}}$$

turned out to be

- appliable to bosons of mass m
- important in the development of Quantum field theory

(also written as $\underbrace{E^2 - p^2}$ $= m^2$) (c=1 units)

$(\underbrace{\frac{E}{c}, p}_{\text{4 vector}})$ or (E, p)

$$\boxed{p^\mu p_\mu = m^2}$$

QM $\begin{cases} \vec{p} \to \frac{\hbar}{i}\vec{\nabla} \to \frac{1}{i}\vec{\nabla} & (\hbar = 1 \text{ units}) \\ E \to i\hbar\frac{\partial}{\partial t} \end{cases}$

But $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$ also form a 4-vector

$$\underbrace{(\partial^\mu \partial_\mu + m^2)\phi(x) = 0}_{\text{Klein Gordon Equation}}$$

or

$$(\Box + m^2)\phi = 0$$

Dirac

- didn't like E^2 and preferred E
- So, he wanted to work on factorizing

$$\underbrace{E^2 - c^2p_x^2 - c^2p_y^2 - c^2p_z^2 - m^2c^4}_{5\ \text{terms}}$$

into something like $(E + \text{something})(E - \text{something})$,

Want to see $(? + m)(? - m)$

then $(? - m)\psi = 0$ becomes his (Dirac) equation.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad [\text{Dirac Equation}]$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \; ; \; \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \qquad k = 1, 2, 3$$

4×4 matrices

But massless ($m=0$) fermion case is simpler!

Massless Fermion

$$E^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2 = 0$$

"How to factorize this expression?"

Recall: Often some "new mathematics" is needed in factorization.

- $x^2 - y^2 = (x+y)(x-y)$
- $\underbrace{x^2 + y^2}_{\text{2 terms}} = (x+iy)(x-iy)$ needs "i"
- $\left.\begin{array}{l} x^2+y^2+z^2 \,? \\ \text{or } \underbrace{p_x^2 + p_y^2 + p_z^2}_{} \,? \end{array}\right\}$ 3 terms

Can factorize it if we extend consideration to 2x2 matrices

- Consider $\sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \vec{\sigma} \cdot \vec{p}$

  where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & p_z \end{pmatrix}$$

$$(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & p_z \end{pmatrix} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & p_z \end{pmatrix} = \begin{pmatrix} p_x^2 + p_y^2 + p_z^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix}$$

$$\therefore \quad E^2 - c^2(p_x^2+p_y^2+p_z^2) = E^2 - c^2(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})$$

$$= \left(E + c(\vec{\sigma}\cdot\vec{p})\right)\left(E - c(\vec{\sigma}\cdot\vec{p})\right)$$

Thus, a relativistic equation for massless fermion is

$$\left(E + c(\vec{\sigma}\cdot\vec{p})\right)\left(E - c(\vec{\sigma}\cdot\vec{p})\right)\varphi = 0$$

where φ is a two-component wavefunction

or simply $\left[E - c(\vec{\sigma}\cdot\vec{p})\right]\varphi = 0$

or $\underbrace{c(\vec{\sigma}\cdot\vec{p})\varphi = E\varphi}_{\text{for massless fermions}}$

Band Structure of Graphene at K turns out to be describable by an equation of this form!